Systems of Equations

There are 3 methods to solving a system of equations:

- Graphing
- Substitution
- Addition/Elimination Method

Solving by Graphing:

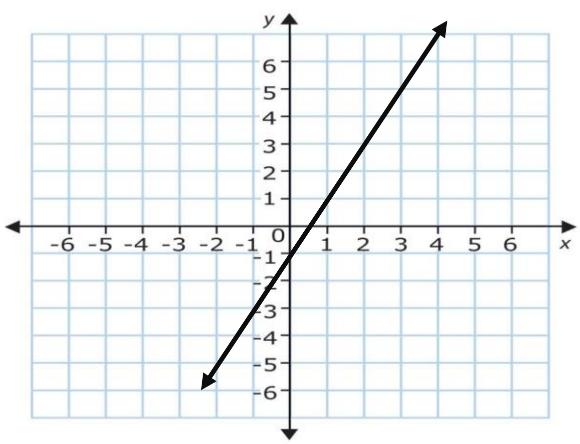
1. Solve the first equation for y -> put it in the form y = mx + b
    a. If the line is already solved for y, skip right down to step 2
2. Graph the line for the first equation
3. Repeat steps 1 and 2 with the second equation
4. The intersection point of the two lines is the solution to the system. If the lines are the same, there are infinitely many solutions. If the lines are parallel and do not intersect, then there are no solutions

Example: Solve the following system of equations by graphing

$y = 2x - 1$

$3y + x = 11$

Since the first equation is solved for y, we are able to graph it right away.



x	y
-2	-5
-1	-3
0	-1
1	1
2	3
3	5

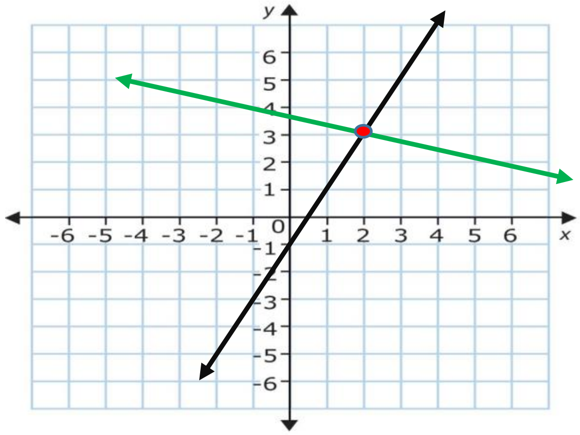
Now we need to solve the second equation for y:

$3y + x = 11$ -> Subtract x from both sides

$3y = -x + 11$ -> Divide both sides by 3

$y = -\frac{1}{3}x + \frac{11}{3}$ or $y = -\frac{1}{3}x + 3\frac{2}{3}$

Now we can graph the second line:



x	y
-2	4.333333
-1	4
0	3.666667
1	3.333333
2	3
3	2.666667

We can see the intersection point is the point (2,3). This is the solution to the system. To check our solution, we plug in the point into our two equations to verify it:

$3 = 2(2) - 1$ $\qquad$ $3(3) + 2 = 11$

$3 = 3$ ✓ $\qquad$ $11 = 11$ ✓

The solution checks out, and thus our solution to the system is (2,3)

Solving by Substitution:

1. Solve ONE of the equations for x or for y
   a. This step can be skipped if one of the equations is already solved for x or y
2. Substitute the result into the other equation for that variable
3. The new equation will now be all x terms, or all y terms. Solve the equation for the variable.
4. Plug the answer back into the original equation (the one you initially solve for x or y) to find the other answer

Example: Solve the following system of equations by substitution

$x = 4y - 9$

$5y + 3x = -10$

Since the first equation is already solved for x, we can substitute right away. We will substitute $4y - 9$ in for x one the second equation, and solve:

$5y + 3(4y - 9) = -10$

$5y + 12y - 27 = -10$

$17y - 27 = -10$

$17y = 17$

$y = 1$

Now use $y = 1$ to find the value for x:

$x = 4(1) - 9$

$x = -5$

To check and verify our solution, we plug our x and y values into the second equation:

$5(1) + 3(-5) = -10$

$5 - 15 = -10$

$-10 = -10$ ✓

The solution checks out, and thus our solution is (-5,1)

Solve the following system by the substitution method:

$3x - 4y = 15$

$4x + 2y = 14$

Neither equation is solved for one variable, so we will need to solve one of the equations for x or y. Let's choose the second equation, and solve it for y.

$4x + 2y = 14$

$2y = -4x + 14$

$y = -2x + 7$

Now we will substitute $-4x + 27$ in for y on the first equation to solve for x:

$3x - 4(-2x + 7) = 15$

$3x + 8x - 28 = 15$

$11x - 28 = 15$

$11x = 43$

$$x = \frac{43}{11}$$

Now we will use $x = \frac{43}{11}$ in the equation we solved for y:

$$y = -2\left(\frac{43}{11}\right) + 7$$

$y = \frac{-86}{11} + 7$

$y = \frac{-86}{11} + \frac{77}{11}$

$y = \frac{-9}{11}$

To check and verify our solution, we plug our x and y values into the first equation:

$3\left(\frac{43}{11}\right) - 4\left(\frac{9}{11}\right) = 15$

$\frac{129}{11} - \frac{-36}{11} = 15$

$\frac{129}{11} + \frac{36}{11} = 15$

$\frac{165}{11} = 15$

$15 = 15$ ✓

The solution checks out, and thus our solution is $(\frac{43}{11}, \frac{-9}{11})$

Solving by the Addition Method:

1. Get the x's and y's on the same side of the equation and lined up with each other.
2. Find a common multiple for EITHER the x's or the y's, whichever you choose to eliminate.
3. Make one of those multiples a negative, so that the equations can be added together.
4. Once the equations are added together, one of the variables will be eliminated. Finish solving for the variable that is still present.
5. Take your answer from #4, and plug it into either of the equations to find the other value.

Example: Solve the system of equations by the addition method

$3x + 7y = -63$

$4x - 3y = -10$

Because the one of the y's is positive and the other is negative, we will eliminate, so that we don't have to multiply by a negative. A common multiple for 7 and 3 is 21.

This means we will multiply the first equation by 3, and the second equation by 7. This will help us solve for x.

$3(3x + 7y = -63)$ -> $9x + 21y = -189$

$7(4x - 3y = -10)$ -> $\underline{28x - 21y = -70}$

$$37x = -259$$

$$x = -7$$

Now, we will plug $x = -7$ into the first equation to solve for y:

$3(-7) + 7y = -63$

$-21 + 7y = -63$

$7y = -42$

$y = -6$

To check and verify our solution, we will plug our x and values into the second equation:

$4(-7) - 3(-6) = -10$

$-28 - (-18) = -10$

$-28 + 18 = -10$

$-10 = -10$ ✓

The solution checks out, and thus our solution is (-7,-6)

Solve the following system of equations by the addition method:

$6x + 5y = 16$

$4x + 9y = 24$

Because there are no negative numbers, we will have to multiply by a negative. Let's choose to eliminate x. A common multiple for 6 and 4 is 12.

This means we will multiply the first equation by 2, and the second equation by -3 (we choose the second equation to be negative):

$2(6x + 5y = 16)$ -> $12x + 10y = 32$

$-3(4x + 9y = 24)$ -> $\underline{-12x - 27y = -72}$

$$-17y = -40$$

$$y = \frac{40}{17}$$

Now we will plug $y = \frac{40}{17}$ into the first equation to solve for x:

$$6x + 5\left(\frac{40}{17}\right) = 16$$

$$6x + \frac{200}{17} = 16$$

$$6x = 16 - \frac{200}{17} = \frac{272}{17} - \frac{200}{17}$$

$$6x = \frac{72}{17}$$

$$x = \frac{72}{102} = \frac{12}{17}$$

To check and verify our solution, we will plug our x and y values into the second equation:

$$4\left(\frac{12}{17}\right) + 9\left(\frac{40}{17}\right) = 24$$

$$\frac{48}{17} + \frac{360}{17} = 24$$

$$\frac{408}{17} = 24$$

$$24 = 24 \checkmark$$

Our solution checks out, and thus our solution is $(\frac{12}{17}, \frac{40}{17})$

When to Use the Different Methods

- Use the graphing method when instructed to graph the equations
- Use the substitution method when one of the equations is already solved for x or y. Also, use the substitution method when one of the coefficients for any of the x's or y's is 1.
- Use the addition method when none of the coefficients are 1. Also, use the addition method when none of the equations are solved for x or y.

Solving a System with Three Equations by Elimination

$$2x + 3y - z = 2$$

$$6x - 5y + 4z = -8$$

$$y - 5z = 19$$

First, we will use the top two equations to eliminate x.

-3$(2x + 3y - z = 2)$	->	$-6x - 9y + 3z = -6$
$6x - 5y + 4z = -8$	->	$\underline{6x - 5y + 4z = -8}$
		$-14y + 7z = -14$

This allows us to use this new equation with the original third equation to eliminate either y or z. We will eliminate y.

$-14y + 7z = -14$	->	$-14y + 7z = -14$	->	$-14y + 7z = -14$
$y - 5z = 19$	->	$14(y - 5z = 19)$	->	$\underline{14y - 70z = 266}$
				$-63z = 252$
				$z = -4$

Now we will use $z = -4$ to help us find y.

$y - 5(-4) = 19$

$y + 20 = 19$

$y = -1$

Now we will use $z = -4$ and $y = -1$ to find x.

$2x + 3(-1) - (-4) = 2$

$2x - 3 + 4 = 2$

$2x = 1$

$$x = \frac{1}{2}$$

So our solution is $x = \frac{1}{2}, y = -1, z = -4$

Solving a System with Three Equations by Row Echelon

$4x - 3y + 5z = 57$

$3x + 2y - 2z = -6$

$6x - 7y - 2z = 24$

First, we need to get the coefficient in front of the x to be a 1. We can do this by taking Row 1 – Row 2.

$4x - 3y + 5z = 57$ R2

$\underline{3x + 2y - 2z = -6}$ R1

$x - 5y + 7z = 63$ R2-R1

This becomes the new first row of the system. Next, we need to get x eliminated from the second equation. We do this by taking $2R_2 - R_3$

$2(3x + 2y - 2y = -6)$	->	$6x + 4y - 4z = -12$	$2R_2$
$6x - 7y - 2z = 24$	->	$6x - 7y - 2z = 24$	R_3
		$11y - 2z = -36$	$2R_2 - R_3$

Next, we need the coefficient in front of that y to be a 1, so we divide everything by 11 and get:

$$y - \frac{2}{11}z = -\frac{36}{11}$$

Lastly, we need to eliminate both x and y for the third equation to make the coefficient in front of the z to be 1. First, we will eliminate x. We will do this by taking $3R_1 - 4R_2$

$3(4x - 3y + 5z = 57)$	->	$12x - 9y + 15z = 171$	$3R_1$
$4(3x + 2y - 2z = -6)$	->	$12x + 8y - 8z = -24$	$4R_2$
		$-17y + 23z = 195$	$3R_1 - 4R_2$

Now, we use this result with the previous result to eliminate y.

$11y - 2z = -36$	->	$17(11y - 2z = -36)$	->	$187y - 34z = -612$
$-17y + 23z = 195$	->	$11(-17y + 23z = 195)$	->	$\underline{-187y + 253z = 2145}$
				$219z = 1533$

$$z = 7$$

Now we have these 3 equations:

$x - 5y + 7z = 63$

$y - \frac{2}{11}z = -\frac{36}{11}$

$z = 7$

We have $z = 7$, and we use that in the second equation.

$y - \frac{2}{11}(7) = -\frac{36}{11}$

$y - \frac{14}{11} = -\frac{36}{11}$

$y = -\frac{22}{11} = -2$

Now use both of our results for y and z to find x using the top equation

$x - 5(-2) + 7(7) = 63$

$x + 10 + 49 = 63$

$x = 4$

Thusly, our solution is $x = 4, y = -2, z = 7$